Session 2 : solution

Exercise 1

$$H = -J \sum_{i=1}^{N-1} s_i s_{i+1}$$

$$\Rightarrow \quad Z_N = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})} = \sum_{\{s_i\}} \prod_{i=1}^{N-1} e^{\beta J s_i s_{i+1}} =$$

for N sites

$$= \sum_{s_N=\pm1} \sum_{s_{N-1}=\pm1} \cdots \sum_{s_1=\pm1} e^{\beta J s_1 s_2} e^{\beta J s_2 s_3} \cdots e^{\beta J s_{N-1} s_N} =$$

$$= \sum_{s_N=\pm1} \cdots \sum_{s_2=\pm1} \left(\sum_{s_1=\pm1} e^{\beta J s_1 s_2} \right) e^{\beta J s_2 s_3} \cdots e^{\beta J s_{N-1} s_N} =$$

$$= \sum_{s_N=\pm1} \cdots \sum_{s_2=\pm1} \left[2 \underbrace{\cosh(\beta J s_2)}_{\substack{\text{even function:} \\ =\cosh(\beta J) \text{ for both values of } s_2}} \right] e^{\beta J s_2 s_3} \cdots e^{\beta J s_{N-1} s_N} =$$

$$= \left[2\cosh(\beta J) \right] \sum_{s_N=\pm1} \cdots \left(\sum_{s_2=\pm1} e^{\beta J s_2 s_3} \right) \cdots e^{\beta J s_{N-1} s_N} =$$

$$= \left[2\cosh(\beta J) \right]^2 \underbrace{\sum_{s_N=\pm1} \cdots \sum_{s_3=\pm1} \prod_{i=3}^{N-1} e^{\beta J s_i s_{i+1}}}_{Z_{N-2}}$$

In general :

$$Z_N = [2\cosh(\beta J)]\, Z_{N-1} = [2\cosh(\beta J)]^N$$

Question : is it ever 0 ~~or~~ ∞ for any β ?

$\Rightarrow$ no phase transition (apart for $\beta = \infty \Rightarrow T = 0$)

$\Rightarrow$ THE 1D ISING MODEL HAS NO PHASE TRANSITION.

Exercise 2

$P(\{s_i\})$

$$F = \overbrace{\sum_{\{s_i\}} \left(-J \sum_{\langle i,j \rangle} s_i s_j P(\{s_i\}) - h \sum_i s_i P(\{s_i\}) \right)}^{U} +$$

$$+ \underbrace{k_B T \sum_{\{s_i\}} P(\{s_i\}) \ln P(\{s_i\})}_{-TS}$$

Mean-field :

$$P(\{s_i\}) = \prod_i p(s_i) \qquad p_i(\cdot) = p(\cdot) \quad \forall i$$

then $$m = \sum_S S \cdot p(S)$$

$$F = -J \sum_{\langle i,j \rangle} \Big[\sum_{s_i = \pm 1} s_i \, p(s_i) \Big] \Big[\sum_{s_j = \pm 1} s_j \, p(s_j) \Big] +$$

$$- h \sum_i \Big[\sum_{s_i = \pm 1} s_i \, p(s_i) \Big] +$$

$$+ k_B T \sum_{\{s_i\}} \prod_j p(s_j) \sum_j \ln p(s_j) =$$

$$= -J \sum_{\langle i,j \rangle} m^2 - h \sum_i m +$$

$$+ k_B T \sum_i \sum_{\{s_i\}} p(s_i) \ln p(s_i)$$

Let's rewrite m

$$m = p(+) - p(-)$$

$p(\ )$ is normalized: $p(+) + p(-) = 1$

$$\begin{cases} p(+) - p(-) = m \\ p(+) + p(-) = 1 \end{cases} \Rightarrow \begin{cases} p(+) = \dfrac{1+m}{2} \\ p(-) = \dfrac{1-m}{2} \end{cases}$$

Then

$$F = -J\frac{z}{2}Nm^2 - hNm + k_BTN \cdot \left(\frac{1+m}{2}\ln\frac{1+m}{2} + \frac{1-m}{2}\ln\frac{1-m}{2} \right)$$

The equilibrium value of ~~m~~ must minimize F

$$\frac{\partial F}{\partial m} = N\left\{-Jzm - h + k_BT\frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)\right\} =$$

$$= Nk_BT\left\{-\beta(Jzm+h) + \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)\right\} = 0$$

Then

$$\frac{1}{2}\ln\frac{1+m}{1-m} = \beta(Jzm+h)$$

$$\Rightarrow \quad \frac{1+m}{1-m} = e^{2\beta(Jzm+h)}$$

$$\Rightarrow \quad 1+m = e^{2\beta(Jzm+h)}(1-m)$$

$$\Rightarrow \quad m\left[1+e^{2\beta(Jzm+h)}\right] = e^{2\beta(Jzm+h)} - 1$$

This translates to

$$m = \frac{e^{2\beta(Jzm+h)} - 1}{e^{2\beta(Jzm+h)} + 1}$$

Dividing above and below by $e^{\beta(Jzm+h)}$

we obtain

$$\boxed{m = \tanh\left[\beta(Jzm+h)\right]}$$